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Uniform q-series asymptotics for staircase polygons

T Prellberg†‡

Department of Mathematics, The University of Melbourne, Parkville, Victoria 3052, Australia

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Abstract. We present a uniform asymptotic expansion for the area-perimeter generating function of staircase polygons by calculating the asymptotic behaviour of the alternating q-series ${}_1\phi_1(0; y; q, x)$ as $q \to 1^-$ from a new integral representation. This leads to a direct calculation of the scaling function for this model.

1. Introduction

Recently, we investigated the tricritical behaviour of geometric cluster models [1]. In this class of models we have the vesicle model of self-avoiding polygons, enumerated by perimeter and area. Interest in this model resulted from the study of models of closed fluctuating membranes (or vesicles). The influence of an osmotic pressure difference on such closed membranes can be studied by a lattice model of closed surfaces. In two dimensions, this is just the geometric model of self-avoiding polygons, which was investigated by Leibler *et al* [2]. Among other parameters, they analysed its scaling behaviour with respect to area and perimeter fugacities. The general features of this model were established by Fisher *et al* [3]. They showed rigorously that the model of self-avoiding polygons on the square lattice $\mathbb{Z}^2$ exhibits the singularity diagram displayed in figure 2, characterized by an essential singularity at unit area fugacity. Moreover, they argued that the same singularity diagram exists for a suitably defined model of closed hypersurfaces in $\mathbb{Z}^d$ in dimension $d > 2$. They also presented numerical estimates for the critical exponents associated with this model.

However, as very little else can be said rigorously about this model, it is desirable to look for simplified models that might be more amenable to rigorous treatment, hopefully without destroying the very transition that one intends to study. For geometric cluster models, it is well known that the introduction of a suitable constraint of (partial) directedness can lead to exact solvability (see e.g. [4]). Introducing such a constraint into the vesicle model of self-avoiding polygons leads to various models of partially convex vesicles [5–7]. These models turn out to be solvable, in the sense that an explicit expression for their generating functions can be given. These expressions can be obtained from recurrence relations or functional equations, and are generally quotients of alternating q-series. However, the lack of suitable asymptotic expansions for these series forced us to use indirect methods to extract the critical behaviour [5–7]. In the discrete case, a perturbation expansion along with a tricritical scaling ansatz led to the computation of the complete set of critical exponents from nonlinear functional equations. Alternatively, we considered, as a further simplification, a semicontinuous version of these models. Here, one can derive nonlinear

† Current address: Department of Mathematics, University of Oslo, POB 1053 Blindern, N-0316 Oslo, Norway.

‡ E-mail address: prel@math.uio.no

differential equations from which we extracted scaling solutions via the method of dominant balance. In particular, it turned out that the semicontinuous versions of the three models of staircase polygons, directed column-convex polygons, and column-convex polygons all share the same exponents and scaling function.

In order to calculate the scaling behaviour for the discrete lattice models, we thus had to either invoke a scaling assumption or a universality argument to bridge the gap. The aim of this paper is to now close this gap by presenting a direct calculation of the asymptotic behaviour of the generating function. Following a suggestion of Philippe Flajolet to Richard Brak [8] we choose to attack the problem by seeking a suitable contour integral representation of the q-series. A study of its saddle-point structure then leads us to a suitable asymptotic expansion via standard techniques. In particular, we come to understand the (mathematical) origin of the singular point in this model: it is caused by the collision of two saddle points.

The structure of this paper is as follows. In the remaining part of this section we briefly introduce the model of staircase polygons and give its generating function. After a brief description of the singularity structure of this generating function we then present our main result and read off the critical exponents and the scaling function for this model. In section 2 we formulate a contour integral representation for the relevant q-Bessel function and in section 3 we derive some asymptotic expressions for the q-products which appear in the representation. In section 4 we then use the contour integral to derive a uniform asymptotic expansion for q-Bessel functions (involving Airy functions) and in section 5 we conclude by applying these techniques to the generating function of staircase polygons. We point out that these techniques can also be applied to other partially convex polygon models. Due to the calculations from the semicontinuous models, we expect to get similar results as for staircase polygons, although the calculations will be more difficult to carry out in detail.

The set of staircase polygons is defined as the set of all polygons on the square lattice whose perimeter consists of two fully directed walks with common start and end points (see figure 1). We denote by $c_m^{n_x,n_y}$ the number of all staircase polygons with $2n_x$ horizontal steps and $2n_y$ vertical steps which enclose an area of size m, and define the polygon-generating function $G(x, y, q)$ to be

$$G(x,y,q) = \sum c_m^{n_x,n_y} x^{n_x} y^{n_y} q^m \,. \tag{1.1}$$

In [5] it was shown that the generating function satisfies the nonlinear functional equation

$$G(x,y,q) = \{G(qx,y,q) + qx\}\{y + G(x,y,q)\} \tag{1.2}$$

from which one can derive an explicit expression

$$G(x,y,q) = y\left(\frac{H(q^2x,qy,q)}{H(qx,y,q)} - 1\right) \qquad \text{with} \qquad H(x,y,q) = \sum_{n=0}^{\infty} \frac{(-x)^n q^{\binom{n}{2}}}{(q;q)_n (y;q)_n} \tag{1.3}$$

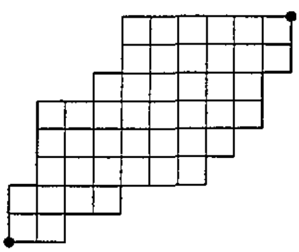

Figure 1. A staircase polygon with width $n_x = 10$, height $n_y = 8$, and area $m = 45$, having weight $x^{10}y^8q^{45}$. The marked sites denote the start and end of the fully directed walks.

where we have used the q-product notation

$$(t;q)_n = \prod_{m=0}^{n-1}(1-tq^m)\,. \tag{1.4}$$

The function $H(x,y,q) = {}_1\phi_1(0;y;q,x)$ is a q-deformation of a Bessel function [9]. One sees that the limit $q \to 1^-$ is singular. However, from the functional equation (1.2) it is clear that in the limit $q=1$ the perimeter generating function $G(x,y)$ is, in fact, algebraic,

$$G(x,y,1) = \frac{1-x-y}{2} - \sqrt{\left(\frac{1-x-y}{2}\right)^2 - xy}\,. \tag{1.5}$$

The singularity structure of this generating function is of particular interest to us. We briefly sketch the form generally expected for polygon models (for details see Brak *et al* [1]). Consider for simplicity the generating function

$$G(x,q) = G(x,x,q) = \sum c_m^n x^n q^m \tag{1.6}$$

where c_m^n is the number of polygons with perimeter $2n$ and area m. The finite-area partition function for polygons with fixed area m is then

$$A_m(x) = \sum_n c_m^n x^n\,. \tag{1.7}$$

$G(x,q)$ is a power series in q with coefficients $A_m(x)$, and its radius of convergence, $q_c(x)$, is given by

$$q_c(x) = \lim_{m\to\infty} A_m(x)^{-1/m}\,. \tag{1.8}$$

A schematic plot of $q_c(x)$ is shown in figure 2. The existence of this limit can be shown using sub-multiplicative inequalities [3]. There exists a critical value x_t such that $q_c(x)=1$ for $0 < x \leqslant x_t$. For $x < x_t$, the generating function has an essential singularity in q at $q=1$. On the line $q=1$, $G(x,1)$ is finite for $x < x_t$ and is singular with an exponent γ_u as x approaches x_t. At x_t, the generating function is singular with an exponent γ_t as $q \to 1$. For $x > x_t$, the generating function has a simple pole as q approaches $q_c(x) < 1$. The point $(1, x_t)$ is an example of a 'tricritical' point [1], and one expects the singular part of the free energy to show a crossover behaviour of the form

$$G^{\mathrm{sing}}(x,q) \sim (1-q)^{-\gamma_t} f\big(\{1-q\}^{-\phi}\{x_t - x\}\big) \tag{1.9}$$

with

$$f(z) \sim \begin{cases} z^{-\gamma_u} & \text{if} \quad z \to \infty \\ 1 & \text{if} \quad z \to 0 \end{cases} \tag{1.10}$$

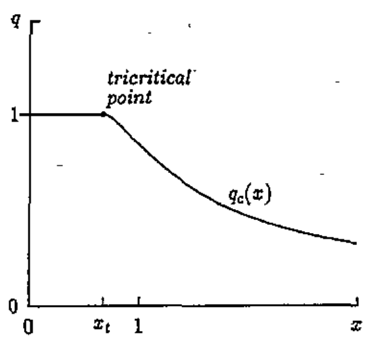

Figure 2. The schematic form of the radius of convergence of the area-perimeter generating function for partially convex vesicle models.

where ϕ is called the tricritical crossover exponent and $\gamma_{\rm u} = \gamma_{\rm t}/\phi$. Here, f and z may be rescaled by non-universal factors, but the scaling function f and the exponents are universal.

More precisely, the scaling function f (if it exists) is defined as a limit where the argument $z = \{1-q\}^{-\phi}\{x_{\rm t} - x\}$ is fixed and $x \to x_{\rm t}$ from the appropriate direction, i.e.

$$f(z) = \lim_{x\to x_{\rm t}} (\{x_{\rm t} - x\}/z)^{\gamma_{\rm u}} G^{\rm sing}\big(x, 1 - (\{x_{\rm t} - x\}/z)^{1/\phi}\big) . \tag{1.11}$$

One sees from this definition that such a scaling function is only defined on approaching the tricritical point, and its validity outside a neighbourhood of that point is not guaranteed.

The calculation presented in this paper will give this scaling function f explicitly. Our main result implies that

$$G(x,y,q) \sim \frac{1-x-y}{2} + \left\{\frac{\mathrm{Ai}'(\alpha\varepsilon^{-2/3})}{\alpha^{1/2}\varepsilon^{-1/3}\mathrm{Ai}(\alpha\varepsilon^{-2/3})}\right\}\sqrt{\left(\frac{1-x-y}{2}\right)^2 - xy} \tag{1.12}$$

as $\varepsilon = -\log q \to 0$, where α is some complicated function of x and y which simplifies to

$$\alpha(x,y) \sim \left(\frac{4}{1-(x-y)^2}\right)^{4/3} \left\{\left(\frac{1-x-y}{2}\right)^2 - xy\right\} \tag{1.13}$$

for $(1-x-y)^2 \approx 4xy$. Moreover, our result is valid *uniformly* for a whole range of x and y as $\varepsilon \to 0$ and not just in the scaling limit which involves the simultaneous limits $\varepsilon \to 0$ and $\alpha \to 0$. This can be easily seen from (1.12), as the factor in the last bracket of (1.12) approaches -1 in the limit $\varepsilon \to 0$, so that we recover (1.5) completely. In order to read off the scaling function more easily, we further restrict ourselves to $x = y$ so that we can write

$$G(x,q) \sim \tfrac{1}{2} - x + 4^{-2/3}\varepsilon^{1/3}\frac{\mathrm{Ai}'(4^{4/3}\{1/4 - x\}\varepsilon^{-2/3})}{\mathrm{Ai}(4^{4/3}\{1/4 - x\}\varepsilon^{-2/3})} . \tag{1.14}$$

Ignoring the non-universal factors, this shows that the scaling function is given as

$$f(z) = -\frac{\mathrm{Ai}'(z)}{\mathrm{Ai}(z)} \tag{1.15}$$

and that

$$\gamma_{\rm u} = -\tfrac{1}{2} \qquad \phi = \tfrac{2}{3} \qquad \gamma_{\rm t} = -\tfrac{1}{3} . \tag{1.16}$$

2. A contour integral representation for q-Bessel series

We are interested in finding a suitable contour integral representation for

$$H(x,y,q) = \sum_{n=0}^{\infty} \frac{(-x)^n q^{\binom{n}{2}}}{(q;q)_n (y;q)_n} . \tag{2.1}$$

For the sake of clarity, we first illustrate our reasoning through some elementary examples. One naturally begins with the standard trick of writing an alternating series as a contour integral which utilizes the fact that the residue of $\pi/\sin(\pi s)$ at integer n is equal to $(-1)^n$. Provided that the coefficients of an alternating power series can be extended to an analytic function in the vicinity of the real axis, one can write

$$\sum_{n=0}^{\infty} (-x)^n c_n = \frac{1}{2\pi \mathrm{i}} \oint_C x^s c(s) \frac{\pi}{\sin(\pi s)}\, \mathrm{d}s \tag{2.2}$$

where the contour is counterclockwise around the zeros of $\sin(\pi s)$. After a suitable deformation of the contour C, this is usually amenable to some sort of saddle-point approximation in the left half-plane. As a simple example, consider the exponential function for negative argument, which we can write as

$$\exp(-x) = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \frac{1}{2\pi \mathrm{i}} \oint_C \frac{x^s}{\Gamma(s+1)} \frac{\pi}{\sin(\pi s)} \, \mathrm{d}s = \frac{1}{2\pi \mathrm{i}} \int_{-c-\mathrm{i}\infty}^{-c+\mathrm{i}\infty} x^s \Gamma(-s) \, \mathrm{d}s \tag{2.3}$$

for some $c > 0$, where we were able to considerably simplify the integrand by using the reflection formula

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)} \,. \tag{2.4}$$

In this example, the poles of the gamma function cancel with the zeros of $\sin(\pi s)$ in the left half-plane, so that that the integrand is, in fact, analytic for $\mathrm{Re}(s) < 0$. This is a considerable simplification, as it reduces the restrictions on possible contour deformations. Naturally, we can use the same trick for, say, the q-exponential, and we get

$$(x;q)_\infty = \sum_{n=0}^{\infty} \frac{(-x)^n q^{\binom{n}{2}}}{(q;q)_n} = -\frac{1}{2\pi \mathrm{i}} \int_{-c-\mathrm{i}\infty}^{-c+\mathrm{i}\infty} \frac{x^s q^{\binom{s}{2}}}{(q;q)_s} \frac{\pi}{\sin(\pi s)} \, \mathrm{d}s \tag{2.5}$$

where we extended the q-product $(x;q)_n$ to complex values via

$$(x;q)_s = \prod_{n=0}^{\infty} \frac{1 - xq^n}{1 - xq^{s+n}} = \frac{(x;q)_\infty}{(xq^s;q)_\infty} \,. \tag{2.6}$$

While this representation can be used, for example, to derive nice transformation formulae [9], it is not that well suited to the derivation of asymptotic expansions. What is required is a suitable q-generalization of (2.4). The q-gamma function is conventionally defined as

$$\Gamma_q(s) = (1-q)^{1-s} (q;q)_{s-1} \tag{2.7}$$

so that in analogy to the product $\Gamma(s)\Gamma(1-s)$ we are led to consider

$$\Lambda_q(s) = (q;q)_{s-1} (q;q)_{-s} \,. \tag{2.8}$$

One readily shows from (2.6) that

$$\Lambda_q(s) = \Lambda_q(1-s) \tag{2.9}$$

$$\Lambda_q(s) = \Lambda_q\left(s + \frac{2\pi \mathrm{i}}{-\log q}\right) \tag{2.10}$$

$$\Lambda_q(s) = -q^{-s} \Lambda_q(s+1) \,. \tag{2.11}$$

$\Lambda_q(s)$ has simple poles at $s = n + m2\pi \mathrm{i}/\log q$ for integer m and n, and a straightforward calculation gives the residues:

$$\mathrm{Res}\left[\Lambda_q(s); s = n + m\frac{2\pi \mathrm{i}}{\log q}\right] = -\frac{(-1)^n q^{\binom{n}{2}}}{\log q} \,. \tag{2.12}$$

$\Lambda_q(s)$ has no zeros, so that its inverse $1/\Lambda_q(s)$ is an entire function. Using Jacobi's triple product identity one can find alternate expressions for $\Lambda_q(s)$ [10], but for our purpose the knowledge of the singularity structure along with the values of the residues suffices.

Using $\Lambda_q(s)$ rather than $\pi/\sin(\pi s)$ provides a much better representation for the q-exponential

$$(x;q)_\infty = -\frac{\log q}{2\pi \mathrm{i}} \oint_C \frac{x^s}{(q;q)_s} \Lambda_q(s) \, \mathrm{d}s = \frac{\log q}{2\pi \mathrm{i}} \oint_C \left(\frac{x}{q}\right)^s (q;q)_{-s-1} \, \mathrm{d}s \,. \tag{2.13}$$

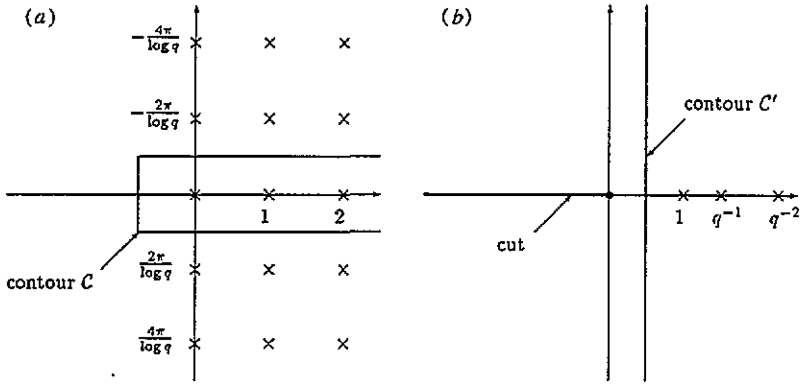

Figure 3. The contours of integration used in (*a*) equation (2.13) and (*b*) equation (2.14). The crosses indicate the poles of the integrand.

The integrand is analytic in the left half-plane, but the contour is restricted by the poles of the integrand at $s = n + m2\pi\mathrm{i}/\log q$ for integer m and non-negative integer n. We choose C to consist of straight-line segments between $-\pi\mathrm{i}/\log q + \infty$, $-\pi\mathrm{i}/\log q - c$, $\pi\mathrm{i}/\log q - c$ and $\pi\mathrm{i}/\log q + \infty$ for some $c > 0$ (see figure 3(*a*)). It is convenient to change the integration variable to $z = q^{-s}$ so that

$$(x;q)_\infty = -\frac{(q;q)_\infty}{2\pi\mathrm{i}} \oint_{C'} \frac{z^{-\log x/\log q}}{(z;q)_\infty}\,\mathrm{d}z\,. \tag{2.14}$$

The integrand has poles at q^{-n} for non-negative integer n and a branch cut from zero to minus infinity. The contour C' now runs from $\mathrm{i}\infty$ to $\mathrm{i}c'$ for $c' = q^{-c}$, then in a semi-circle centred at zero to $-\mathrm{i}c'$ and further to $-\mathrm{i}\infty$ and can easily be transformed to run along the straight line $(\rho+\mathrm{i}\infty, \rho-\mathrm{i}\infty)$ for some $0 < \rho < 1$ (see figure 3(*b*)). With these introductory remarks we have motivated our first lemma.

Lemma 2.1. For complex x with $|\arg(x)| < \pi$ and $0 < q < 1$ we have

$$\frac{(x;q)_\infty}{(q;q)_\infty} = \frac{1}{2\pi\mathrm{i}} \int_{\rho-\mathrm{i}\infty}^{\rho+\mathrm{i}\infty} \frac{z^{-\log x/\log q}}{(z;q)_\infty}\,\mathrm{d}z \qquad 0 < \rho < 1\,. \tag{2.15}$$

Proof. Define contours $C_N = C_N^1 \cup C_N^2 \cup C^3$, where $C_N^1 = \{q^{-N-1/2}\exp(\mathrm{i}t) : |t| < \pi/2\}$, $C_N^2 = \{t : q^{1/2} \leqslant |t| \leqslant q^{-N-1/2}\}$, and $C^3 = \{q^{1/2}\exp(\mathrm{i}t) : |t| < \pi/2\}$, and integrate anti-clockwise over C_N to get

$$\frac{1}{2\pi\mathrm{i}} \oint_{C_N} \frac{z^{-\log x/\log q}}{(z;q)_\infty}\,\mathrm{d}z = \sum_{n=0}^{N} \mathrm{Res}\left[\frac{z^{-\log x/\log q}}{(z;q)_\infty}; z = q^{-n}\right] = -\sum_{n=0}^{N} \frac{(-x)^n q^{\binom{n}{2}}}{(q;q)_n (q;q)_\infty}\,. \tag{2.16}$$

In order to estimate the contribution from C_N^1, we first need a bound on the integrand for large $|z|$ with $|\arg(z)| < \pi$:

$$\sup_{|z|=q^{-N-1/2}} \left|\frac{z^{-\log x/\log q}}{(z;q)_\infty}\right| \leqslant \frac{|x|^{N+1/2}\exp\left(\frac{\pi^2}{-\log q}\right)}{(q^{1/2};q)_\infty \prod_{n=0}^{N}(q^{-n-1/2}-1)} = \mathrm{O}\left(q^{N^2/2}\right) \tag{2.17}$$

as it is dominated by the product in the denominator. Therefore we have

$$\left|\oint_{C_N^1} \frac{z^{-\log x/\log q}}{(z;q)_\infty}\,\mathrm{d}z\right| \leqslant \pi q^{-N-1/2} \sup_{|z|=q^{-N-1/2}} \left|\frac{z^{-\log x/\log q}}{(z;q)_\infty}\right| = \mathrm{O}\left(q^{N^2/2}\right). \tag{2.18}$$

Now we can write

$$\sum_{n=0}^{\infty}\frac{(-x)^n q^{\binom{n}{2}}}{(q;q)_n(q;q)_\infty} = -\lim_{N\to\infty}\frac{1}{2\pi\mathrm{i}}\oint_{C_N}\frac{z^{-\log x/\log q}}{(z;q)_\infty}\,\mathrm{d}z$$
$$= \frac{1}{2\pi\mathrm{i}}\left\{\int_{-\mathrm{i}\infty}^{-\mathrm{i}q^{1/2}} + \int_{C^3} + \int_{\mathrm{i}q^{1/2}}^{\mathrm{i}\infty}\right\}\frac{z^{-\log x/\log q}}{(z;q)_\infty}\,\mathrm{d}z \tag{2.19}$$

where in the last step we changed the direction of integration. As we have shown that the integrand decays to zero at infinity we can now deform the contour to $(\rho - \mathrm{i}\infty, \rho + \mathrm{i}\infty)$. This concludes the proof of the lemma. □

For $x = q$ we get the identity

$$\frac{1}{2\pi\mathrm{i}}\int_{\rho-\mathrm{i}\infty}^{\rho+\mathrm{i}\infty}\frac{\mathrm{d}z}{z(z;q)_\infty} = 1 \tag{2.20}$$

where the integral is independent of q. This identity can be understood as follows. We transform the contour to a circle around the origin, so that $|z| < 1$. Now, we are allowed to expand $(z;q)^{-1} = \sum_{n=0}^{\infty} z^n/(q;q)_n$ and upon exchanging integration and summation we observe that only the term $n = 0$ with a residue of 1 contributes to the integral.

As the proof shows, it is of course not necessary to proceed via the function $\Lambda_q(s)$ to derive the contour integral representation (2.15). In hindsight, there would have been a more direct way to write down the contour integral representation for the q-exponential function by simply observing that $(z;q)_\infty^{-1}$ has poles at $z = q^{-n}$ for non-negative integer n with residues

$$\mathrm{Res}\big[(z;q)_\infty^{-1}; z = q^{-n}\big] = -\frac{(-1)^n q^{\binom{n}{2}}}{(q;q)_n(q;q)_\infty} \tag{2.21}$$

and that these residues already contain much of the structure of the coefficients of the series. Generalizing the above to $H(x, y, q)$ is now immediate.

Lemma 2.2. For complex x with $|\arg(x)| < \pi$, complex y with $y \neq q^{-n}$ for non-negative integer n, and $0 < q < 1$, we have

$$H(x,y,q) = \frac{1}{2\pi\mathrm{i}}\frac{(q;q)_\infty}{(y;q)_\infty}\int_{\rho-\mathrm{i}\infty}^{\rho+\mathrm{i}\infty}\frac{(y/z;q)_\infty}{(z;q)_\infty}z^{-\log x/\log q}\,\mathrm{d}z \qquad 0 < \rho < 1\,. \tag{2.22}$$

Proof. We choose contours C_N^1, C_N^2, C^3 and C_N as in the proof of lemma 2.1 and show that the contribution from the integral over C_N^1 tends to zero. For this, we estimate the integrand as

$$\sup_{|z|=q^{-N-1/2}}\left|z^{-\log x/\log q}\frac{(y/z;q)_\infty}{(z;q)_\infty}\right| \leqslant \frac{|x|^{N+1/2}\exp\left(\frac{\pi^2}{-\log q}\right)(-|y|q^{N+1/2};q)_\infty}{(q^{1/2};q)_\infty\prod_{n=0}^{N}(q^{-n-1/2}-1)}$$
$$= \mathrm{O}\left(q^{N^2/2}\right). \tag{2.23}$$

Now, integration over C_N by computing the residues at q^{-n} and taking the limit $N \to \infty$ completes the proof. □

We now have a suitable representation of H to consider the $q \to 1^-$ asymptotics.

3. Asymptotics of q-products

In order to get an asymptotic expansion from the contour integrals in the previous section we first need to derive some asymptotic expansions for the q-products which appear in the integrand. As these q-products are basically q-deformations of the gamma function, one needs to derive a q-analogue of Stirling's formula [11]. For a heuristic derivation, we simply take the logarithm and expand

$$\log(t;q)_\infty = \sum_{n=0}^{\infty}\log(1-tq^n) = -\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{(tq^n)^m}{m} = -\sum_{m=1}^{\infty}\frac{1}{m}\frac{t^m}{1-q^m}$$
$$\sim \frac{1}{\log q}\mathrm{Li}_2(t) + \tfrac{1}{2}\log(1-t) + \sum_{n=1}^{\infty}\frac{B_{2n}}{(2n)!}(\log q)^{2n-1}\left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)^{2n-2}\frac{t}{1-t} \tag{3.1}$$

where in the last step we have used the expansion $t/(\mathrm{e}^t-1) = \sum_{n=0}^{\infty}(B_n/n!)t^n$, where B_n are the Bernoulli numbers. Here $\mathrm{Li}_2(t) = \sum_{m=1}^{\infty} t^m/m^2$ is the Euler dilogarithm [12] which can be extended to complex t by the integral representation

$$\mathrm{Li}_2(t) = -\int_0^t \frac{\log(1-u)}{u}\,\mathrm{d}u\,. \tag{3.2}$$

Later, we will use the functional equation

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(1-x) = \tfrac{1}{6}\pi^2 - \log(x)\log(1-x) \tag{3.3}$$

which is valid for $0 \leqslant x \leqslant 1$. We want to show that the expansion (3.1) is in fact an asymptotic expansion for $q \to 1^-$ uniform in t for some complex domain. To formalize this, we need to resort to the Euler–Maclaurin summation formula [13], which we state here for completeness sake.

Lemma 3.1 (Euler–Maclaurin). If $f \in C^{2m}[0,N]$ for integer N then

$$\sum_{n=0}^{N} f(n) = \int_0^N f(x)\,\mathrm{d}x + \tfrac{1}{2}[f(0)+f(N)] + \sum_{n=1}^{m-1}\frac{B_{2n}}{(2n)!}\left(f^{(2n-1)}(N) - f^{(2n-1)}(0)\right) + R_m \tag{3.4}$$

where

$$R_m = \int_0^N \frac{B_{2m} - B_{2m}(x-[x])}{(2m)!} f^{(2m)}(x)\,\mathrm{d}x\,. \tag{3.5}$$

Here $B_n(x)$ is the nth Bernoulli polynomial, $B_n = B_n(0)$, and m is any positive integer.

We now apply this lemma to $\log(t;q)_\infty$ with $f(x) = \log(1-tq^x)$. For $0<t<1$ this has been done in [11]. Generalizing these results we now prove:

Lemma 3.2. For complex t such that $|\arg(1-t)| < \pi$ and $0<q<1$

$$\log(t;q)_\infty \sim \frac{1}{\log q}\mathrm{Li}_2(t) + \tfrac{1}{2}\log(1-t) + \sum_{n=1}^{\infty}\frac{B_{2n}}{(2n)!}(\log q)^{2n-1}\left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)^{2n-2}\frac{t}{1-t} \tag{3.6}$$

is an asymptotic expansion as $q \to 1^-$. This expansion is uniform for t in any compact domain such that $|\arg(1-t)| < \pi$.

Proof. We use lemma 3.1 with $f(x) = \log(1-tq^x)$. First, we need a formula for its derivatives. We write

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \log(1-tq^x) = -(\log q)^n\left(u\frac{\mathrm{d}}{\mathrm{d}u}\right)^{n-1}\frac{u}{1-u}\bigg|_{u=tq^x}. \tag{3.7}$$

Inserting this into (3.4) we write

$$\log(t;q)_\infty = \int_0^\infty \log(1-tq^x)\,\mathrm{d}x + \tfrac{1}{2}\log(1-t) + \sum_{n=1}^{m-1}\frac{B_{2n}}{(2n)!}(\log q)^{2n-1}\left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)^{2n-2}\frac{u}{1-u}\bigg|_{u=t} + R_m \tag{3.8}$$

$$= \frac{1}{\log q}\mathrm{Li}_2(t) + \tfrac{1}{2}\log(1-t) + \sum_{n=1}^{m-1}\frac{B_{2n}}{(2n)!}(\log q)^{2n-1}\left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)^{2n-2}\frac{t}{1-t} + R_m\,. \tag{3.9}$$

We still have to consider the remainder term

$$R_m = (\log q)^{2m}\int_0^\infty \frac{B_{2m}(x-[x])-B_{2m}}{(2m)!}\left(u\frac{\mathrm{d}}{\mathrm{d}u}\right)^{2m-1}\frac{u}{1-u}\bigg|_{u=tq^x}\mathrm{d}x$$
$$= \frac{(\log q)^{2m-1}}{(2m)!}\int_0^1\left\{B_{2m}-B_{2m}\left(\frac{\log u}{\log q}-\left[\frac{\log u}{\log q}\right]\right)\right\}\left\{\left(u\frac{\mathrm{d}}{\mathrm{d}u}\right)^{2m-1}\frac{tu}{1-tu}\right\}\frac{\mathrm{d}u}{u} \tag{3.10}$$

which we estimate as

$$|R_m| \leqslant |\log q|^{2m-1}\frac{2|B_{2m}|}{(2m)!}\int_0^1\left|\left(u\frac{\mathrm{d}}{\mathrm{d}u}\right)^{2m-1}\frac{tu}{1-tu}\right|\frac{\mathrm{d}u}{u}\,. \tag{3.11}$$

Using lemma 1 in [11] we can write for $n \geqslant 1$

$$\left(u\frac{\mathrm{d}}{\mathrm{d}u}\right)^n\frac{u}{1-u} = \frac{u}{(1-u)^{n+1}}P_{n-1}(u) \tag{3.12}$$

where $P_n(x)$ is a polynomial of degree n with positive integer coefficients satisfying $P_n(0)=1$ and $P_n(1)=(n+1)!$. Therefore,

$$|R_m| \leqslant |\log q|^{2m-1}\frac{2|B_{2m}|}{(2m)!}\int_0^1\frac{|t|u}{|1-tu|^{2m}}P_{2m-2}(|t|u)\frac{\mathrm{d}u}{u}$$
$$\leqslant |\log q|^{2m-1}\frac{|B_{2m}|}{m}\int_0^1\frac{|t|^{2m-1}u^{2m-2}\mathrm{d}u}{|1-tu|^{2m}}$$
$$= |\log q|^{2m-1}\frac{|B_{2m}|}{m}\int_0^{|t|}\frac{x^{2m-1}\,\mathrm{d}x}{|1-\mathrm{e}^{\mathrm{i}\phi}x|^{2m}} \tag{3.13}$$

where $\phi = \arg(t)$. Now we see that, if the integration path has distance ϵ from 1 and t is bounded, then the integral is uniformly bounded. This is certainly the case if we choose t to be in any compact domain such that $|\arg(1-t)| < \pi$. Here $R_m = \mathrm{O}((\log q)^{2m-1})$ uniformly which concludes the proof of the lemma. □

We can use (3.13) to get more explicit error bounds. For instance, if we set $m=1$ we can evaluate the integral explicitly to get

$$\log(t;q)_\infty = \frac{1}{\log q}\mathrm{Li}_2(t) + \tfrac{1}{2}\log(1-t) + \log(q)R(t,q) \tag{3.14}$$

where $R(t,q)$ has a bound independent of q

$$|R(t,q)| \leqslant \tfrac{1}{6}\left(\log|1-t| + \frac{\mathrm{Re}(t)}{\mathrm{Im}(t)}\tan^{-1}\frac{\mathrm{Im}(t)}{1-\mathrm{Re}(t)}\right). \tag{3.15}$$

Clearly, lemma 3.2 does not cover the case $t = q$, which we therefore have to treat separately. There is the beautiful conjugate modulus transformation which we can utilize (see e.g. Hardy's book on Ramanujan [14]):

Lemma 3.3.

$$(q;q)_\infty = \frac{(r/q)^{1/24}}{(r;r)_\infty}\sqrt{\frac{2\pi}{-\log q}} \tag{3.16}$$

where $r = \exp\left(-\frac{4\pi^2}{-\log q}\right)$ and $0 < q < 1$.

Using Poisson summation, a similar formula has been arrived at in [11], where one finds the expression

$$(q;q)_\infty = (r/q)^{1/24} \sum_{n=-\infty}^{\infty} \left(r^{n(6n+1)} - r^{(3n+1)(2n+1)}\right)\sqrt{\frac{2\pi}{-\log q}} \tag{3.17}$$

without the identification of the sum with $(r;r)_\infty^{-1}$.

Therefore, to leading order we have

$$\log(q;q)_\infty = \frac{\pi^2}{6\log q} + \tfrac{1}{2}\log\frac{2\pi}{-\log q} + \mathrm{O}(\log q)\,. \tag{3.18}$$

4. Uniform asymptotics for the q-Bessel function

We now return to the investigation of $H(x, y, q)$. In this section we provide an asymptotic expansion via its contour integral representation (2.22). We restrict ourselves to $0 < x, y, q < 1$ and introduce the notation $\varepsilon = -\log q$.

First, we need to approximate the integral representation from lemma 2.2 to make an analysis tractable. Using the asymptotic formulae from the previous section we get

Lemma 4.1. Let $0 < x, y < 1$ and $q = \mathrm{e}^{-\varepsilon}$ for $\varepsilon > 0$. Then

$$\begin{aligned} H(x,y,q) = {} & \frac{1}{2\pi\mathrm{i}}\int_{\rho-\mathrm{i}\infty}^{\rho+\mathrm{i}\infty} \mathrm{e}^{\frac{1}{\varepsilon}\{\log(z)\log(x)+\mathrm{Li}_2(z)-\mathrm{Li}_2(y/z)\}}\sqrt{\frac{1-y/z}{1-z}}\,\mathrm{d}z \\ & \times \mathrm{e}^{\frac{1}{\varepsilon}\{\mathrm{Li}_2(y)-\pi^2/6\}}\sqrt{\frac{2\pi}{\varepsilon(1-y)}}\,\{1+\mathrm{O}(\varepsilon)\} \end{aligned} \tag{4.1}$$

where $y < \rho < 1$.

Proof. From lemma 2.2 we have that

$$H(x,y,q) = \frac{1}{2\pi\mathrm{i}}\frac{(q;q)_\infty}{(y;q)_\infty}\int_{\rho-\mathrm{i}\infty}^{\rho+\mathrm{i}\infty}\frac{(y/z;q)_\infty}{(z;q)_\infty}z^{-\log x/\log q}\,\mathrm{d}z \qquad 0<\rho<1\,. \tag{4.2}$$

We can now apply (3.14) to get

$$\frac{(y/z;q)_\infty}{(z;q)_\infty}z^{-\log x/\log q} = \mathrm{e}^{\frac{1}{\varepsilon}\{\log(z)\log(x)+\mathrm{Li}_2(z)-\mathrm{Li}_2(y/z)\}}\sqrt{\frac{1-y/z}{1-z}}\,\mathrm{e}^{\varepsilon\{R(z,q)-R(y/z,q)\}}\,. \tag{4.3}$$

Now if we write $z = \rho + \mathrm{i}t$ and choose $y < \rho < 1$ then (3.15) implies that

$$|R(z,q) - R(y/z,q)| = \mathrm{O}(\max\{\log|t|, 1\}) \tag{4.4}$$

which is not a uniform bound independent of t so that we have to exercise some care. Now, expanding the last exponential term into its power series and exchanging the summation over powers of $\varepsilon\{(R(z,q)-R(y/z,q)\}$ with integration we get an asymptotic series in ε, so that we can write

$$\int_{\rho-\mathrm{i}\infty}^{\rho+\mathrm{i}\infty}\frac{(y/z;q)_\infty}{(z;q)_\infty}z^{-\log x/\log q}\,\mathrm{d}z=\int_{\rho-\mathrm{i}\infty}^{\rho+\mathrm{i}\infty}\mathrm{e}^{\frac{1}{\varepsilon}\{\log(z)\log(x)+\mathrm{Li}_2(z)-\mathrm{Li}_2(y/z)\}}\sqrt{\frac{1-y/z}{1-z}}\,\mathrm{d}z$$
$$+\,\mathrm{O}\left(\varepsilon\int_{\rho-\mathrm{i}\infty}^{\rho+\mathrm{i}\infty}\mathrm{e}^{\frac{1}{\varepsilon}\{\log(z)\log(x)+\mathrm{Li}_2(z)-\mathrm{Li}_2(y/z)\}}\sqrt{\frac{1-y/z}{1-z}}\{R(z,q)-R(y/z,q)\}\mathrm{d}z\right). \tag{4.5}$$

Now we are left with Laplace-type integrals of the form

$$I_n=\int_C \mathrm{e}^{g(z)/\varepsilon}f(z)[h(z,\varepsilon)]^n\,\mathrm{d}z \tag{4.6}$$

where $h(z,\varepsilon)=R(z,q)-R(y/z,q)$, and

$$g(z)=\log(z)\log(x)+\mathrm{Li}_2(z)-\mathrm{Li}_2(y/z)\,. \tag{4.7}$$

Applying the saddle-point method to the $\varepsilon\to 0$ limit, we see from the derivative

$$g'(z)=\frac{1}{z}\log\frac{x}{(1-z)(1-y/z)} \tag{4.8}$$

that there are two saddle points. Around the saddle points $R(z,q)-R(y/z,q)$ is bounded so that in (4.5) the second integral is of the same order of magnitude as the first integral. Therefore we arrive at

$$\int_{\rho-\mathrm{i}\infty}^{\rho+\mathrm{i}\infty}\frac{(y/z;q)_\infty}{(z;q)_\infty}z^{-\log x/\log q}\,\mathrm{d}z$$
$$=\int_{\rho-\mathrm{i}\infty}^{\rho+\mathrm{i}\infty}\mathrm{e}^{\frac{1}{\varepsilon}\{\log(z)\log(x)+\mathrm{Li}_2(z)-\mathrm{Li}_2(y/z)\}}\sqrt{\frac{1-y/z}{1-z}}\,\mathrm{d}z\,\{1+\mathrm{O}(\varepsilon)\}\,. \tag{4.9}$$

Finally we apply (3.14) and (3.18) to the remaining prefactors picking up further multiplicative error terms $\{1+\mathrm{O}(\varepsilon)\}$. This completes the proof. □

Now that we have an asymptotic representation of $H(x,y,q)$ as a genuine Laplace-type integral, we can proceed with the actual calculation of the dominant asymptotic form. The two saddle points are the zeros $z_{1,2}$ of the quadratic equation

$$(z-1)(z-y)+zx=0\,. \tag{4.10}$$

There will be a change in the asymptotic behaviour when the saddles coalesce due to the discriminant changing sign. For the polygon problem the point of coalescence is the tricritical point. Thus to obtain the scaling function we need an expansion that is uniform in both saddle points.

The problem of deriving a uniform asymptotic expansion for the case of two coalescing saddle points has been investigated in [15, 16]. We briefly summarize their analysis here (for a very readable account see [17]). Assume that the two functions f and g are analytic in z in some domain containing a path C, and consider

$$I(\varepsilon;d)=\int_C \mathrm{e}^{\frac{1}{\varepsilon}g(z;d)}f(z)\,\mathrm{d}z\,. \tag{4.11}$$

Moreover, assume that g is analytic in d and that g has two distinct saddle points of multiplicity 1 for $d\neq 0$ which coalesce when $d=0$. We first reparametrize locally by a cubic

$$g(z)=\tfrac{1}{3}u^3-\alpha u+\beta \tag{4.12}$$

so that the saddle points of both expressions coincide. Differentiating (4.12) we get $g'(z)\frac{dz}{du} = u^2 - \alpha$ so that we identify

$$u = \pm\alpha^{1/2} \longleftrightarrow z = z_{1,2} \tag{4.13}$$

and determine α and β from

$$g(z_1) = -\tfrac{2}{3}\alpha^{3/2} + \beta \qquad g(z_2) = \tfrac{2}{3}\alpha^{3/2} + \beta\,. \tag{4.14}$$

Selecting the correct branch of the cubic equation, the transformation given by (4.12) with (4.14) is one-to-one and analytic in a neighbourhood of $d = 0$. In practice, one still has to show additionally that this neighbourhood extends to a domain containing the path of integration. Assuming this to be the case, we proceed by expanding

$$f(z)\frac{dz}{du} = \sum_{n=0}^{\infty}(p_m + q_m u)\left(u^2 - d\right)^m. \tag{4.15}$$

Denoting the image of C as C' and writing

$$V(\lambda) = \frac{1}{2\pi i}\int_{C'} e^{u^3/3-\lambda u}\, du \tag{4.16}$$

we get the asymptotic expansion

$$e^{-\beta/\varepsilon} I(\varepsilon; d) \sim \varepsilon^{1/3} V\left(\alpha\varepsilon^{-2/3}\right)\sum_{m=0}^{\infty} a_m\varepsilon^m + \varepsilon^{2/3} V'\left(\alpha\varepsilon^{-2/3}\right)\sum_{m=0}^{\infty} b_m\varepsilon^m\,. \tag{4.17}$$

The function $V(\lambda)$ is expressible using the Airy function $\mathrm{Ai}(\lambda)$, the exact relation depending on the contour C'. We finish this section by presenting explicit formulae for the coefficients $a_0 = p_0$ and $b_0 = q_0$ of the leading-order terms. Differentiating (4.12) twice and inserting the saddle-point values we get

$$\left.\frac{dz}{du}\right|_{z=z_{1,2},u=\pm\alpha^{1/2}} = \sqrt{\frac{2\alpha^{1/2}}{\pm g''(z_{1,2})}} \tag{4.18}$$

so that we get

$$p_0 + q_0\alpha^{1/2} = f(z_1)\sqrt{\frac{2\alpha^{1/2}}{g''(z_1)}} \qquad \text{and} \qquad p_0 - q_0\alpha^{1/2} = f(z_2)\sqrt{\frac{2\alpha^{1/2}}{-g''(z_2)}}\,. \tag{4.19}$$

We are now in a position to compute the leading asymptotic behaviour of our contour integral.

Lemma 4.2. Let $0 < x, y < 1$ and $q = e^{-\varepsilon}$ for $\varepsilon > 0$. Then

$$\frac{1}{2\pi i}\int_{\rho-i\infty}^{\rho+i\infty} e^{\frac{1}{\varepsilon}(\log(z)\log(x)+\mathrm{Li}_2(z)-\mathrm{Li}_2(y/z))}\sqrt{\frac{1-y/z}{1-z}}\, dz$$
$$= e^{\log(x)\log(y)/2\varepsilon}\left\{p_0\varepsilon^{1/3}\mathrm{Ai}\left(\alpha\varepsilon^{-2/3}\right) + q_0\varepsilon^{2/3}\mathrm{Ai}'\left(\alpha\varepsilon^{-2/3}\right)\right\}\{1 + O(\varepsilon)\} \tag{4.20}$$

where

$$\tfrac{4}{3}\alpha^{3/2} = \log(x)\log\left(\frac{z_m - \sqrt{d}}{z_m + \sqrt{d}}\right) + 2\,\mathrm{Li}_2(z_m - \sqrt{d}) - 2\,\mathrm{Li}_2(z_m + \sqrt{d}) \tag{4.21}$$

with

$$z_{1,2} = z_m \pm \sqrt{d} \qquad z_m = \frac{1+y-x}{2} \qquad \text{and} \qquad d = z_m^2 - y \tag{4.22}$$

and

$$p_0 = \left(\frac{\alpha}{d}\right)^{1/4}(1-x-y) \qquad q_0 = \left(\frac{d}{\alpha}\right)^{1/4}. \tag{4.23}$$

Proof. The exponential part of the integrand

$$g(z) = \log(z)\log(x) + \mathrm{Li}_2(z) - \mathrm{Li}_2(y/z) \tag{4.24}$$

has two saddle points at $z_{1,2}$, where

$$z_{1,2} = z_m \pm \sqrt{d} \qquad z_m = \frac{1+y-x}{2} \qquad \text{and} \qquad d = z_m^2 - y\,. \tag{4.25}$$

Using the reparametrization (4.12) we determine the constants α and β from (4.14) as

$$\tfrac{4}{3}\alpha^{3/2} = \log(x)\log\left(\frac{z_m - \sqrt{d}}{z_m + \sqrt{d}}\right) + 2\mathrm{Li}_2(z_m - \sqrt{d}) - 2\mathrm{Li}_2(z_m + \sqrt{d}) \tag{4.26}$$

$$2\beta = \log(x)\log(y)\,. \tag{4.27}$$

We need to show that the transformation (4.12) extends to a domain containing the contour $\mathcal{C} = \{\rho + \mathrm{i}t;\ -\infty < t < \infty\}$. This can be done by explicitly computing the relevant branch of $u(z)$ (i.e. the branch that is real for z real)

$$u(z) = \left(\tfrac{3}{2}(g(z)-\beta) + \sqrt{\left(\tfrac{3}{2}(g(z)-\beta)\right)^2 - \alpha^3}\right)^{1/3} \tag{4.28}$$

$$+\alpha\left(\tfrac{3}{2}(g(z)-\beta) + \sqrt{\left(\tfrac{3}{2}(g(z)-\beta)\right)^2 - \alpha^3}\right)^{-1/3}. \tag{4.29}$$

One sees on closer inspection that the mapping is indeed one-to-one in a domain containing $z = \rho + \mathrm{i}t$ with real t.

For $t \to \pm\infty$ the asymptotic behaviour of $g(z)$ is dominated by the dilogarithm

$$g(\rho + \mathrm{i}t) \sim \mathrm{Li}_2(\rho + \mathrm{i}t) \sim -\tfrac{1}{2}\log^2|t| \pm \mathrm{i}\tfrac{1}{2}\pi\log|t| \qquad t \to \pm\infty \tag{4.30}$$

so that the asymptotic behaviour of $u(z)$ is given by

$$u(\rho + \mathrm{i}t) \sim \exp(\pm\mathrm{i}\pi/3)\left(\tfrac{3}{2}\log^2|t|\right)^{1/3} \qquad t \to \pm\infty\,. \tag{4.31}$$

Thus, the path $\mathcal{C}'$ runs from $\infty\mathrm{e}^{-\mathrm{i}\pi/3}$ via the origin to $\infty\mathrm{e}^{+\mathrm{i}\pi/3}$, whence $V(\lambda)$ is, in fact, equal to the Airy function $\mathrm{Ai}(\lambda)$. Finally, we compute the prefactors p_0 and q_0 from (4.19) as

$$p_0 = \left(\frac{\alpha}{d}\right)^{1/4}(1-x-y) \qquad q_0 = \left(\frac{d}{\alpha}\right)^{1/4}. \tag{4.32}$$

Inserting all of this into (4.17) gives (4.20). This completes the proof. □

We can use the functional equation (3.3) for the dilogarithm to write

$$\begin{aligned}\tfrac{4}{3}\alpha^{3/2} &= \log(z_m + \sqrt{d})\log(1 - z_m + \sqrt{d}) - \log(z_m - \sqrt{d})\log(1 - z_m - \sqrt{d}) \\ &\quad + \mathrm{Li}_2(z_m - \sqrt{d}) + \mathrm{Li}_2(1 - z_m - \sqrt{d}) - \mathrm{Li}_2(z_m + \sqrt{d}) - \mathrm{Li}_2(1 - z_m + \sqrt{d})\,.\end{aligned} \tag{4.33}$$

Using the fact that exchanging x and y transforms z_m into $1 - z_m$ and leaves d invariant shows now that α is symmetric in x and y. Therefore, the terms in the asymptotic expansion (4.20) are symmetric in x and y, as they should be. Moreover, in the limit of small d we can expand

$$\alpha^{3/2} = \frac{1}{z_m^2(1-z_m)^2} d^{3/2}\{1 + \mathrm{O}(d)\} \tag{4.34}$$

so that α is basically just a suitable rescaling of d. Combining lemmas 4.1 and 4.2 we finally arrive at the main result of this section.

Lemma 4.3. Let $0 < x, y < 1$ and $q = \mathrm{e}^{-\varepsilon}$ for $\varepsilon > 0$. Then

$$H(x,y,q) = \left\{p_0\varepsilon^{1/3}\mathrm{Ai}\left(\alpha\varepsilon^{-2/3}\right) + q_0\varepsilon^{2/3}\mathrm{Ai}'\left(\alpha\varepsilon^{-2/3}\right)\right\}$$
$$\times \mathrm{e}^{\frac{1}{\varepsilon}\{\mathrm{Li}_2(y)-\frac{1}{6}\pi^2+\log(x)\log(y)/2\}}\sqrt{\frac{2\pi}{\varepsilon(1-y)}}\{1+\mathrm{O}(\varepsilon)\} \tag{4.35}$$

where

$$\tfrac{4}{3}\alpha^{3/2} = \log(x)\log\left(\frac{z_m-\sqrt{d}}{z_m+\sqrt{d}}\right) + 2\,\mathrm{Li}_2(z_m-\sqrt{d}) - 2\,\mathrm{Li}_2(z_m+\sqrt{d}) \tag{4.36}$$

with

$$z_{1,2} = z_m \pm \sqrt{d} \qquad z_m = \frac{1+y-x}{2} \qquad \text{and} \qquad d = z_m^2 - y \tag{4.37}$$

and

$$p_0 = \left(\frac{\alpha}{d}\right)^{1/4}(1-x-y) \qquad q_0 = \left(\frac{d}{\alpha}\right)^{1/4}. \tag{4.38}$$

5. Asymptotics for staircase polygons

Using the contour integral representation (2.22) we now present the contour integral representation for the staircase-generating function.

Lemma 5.1. For complex x with $|\arg(x)| < \pi$, complex y with $y \neq q^{-n}$ for non-negative integer n, and $0 < q < 1$ we have

$$G(x,y,q) = \frac{\int_{\rho-\mathrm{i}\infty}^{\rho+\mathrm{i}\infty}\frac{y(1-z)}{z(z-y)}\frac{(y/z;q)_\infty}{(z;q)_\infty}z^{-\log x/\log q}\,\mathrm{d}z}{\int_{\rho-\mathrm{i}\infty}^{\rho+\mathrm{i}\infty}\frac{1}{(z-y)}\frac{(y/z;q)_\infty}{(z;q)_\infty}z^{-\log x/\log q}\,\mathrm{d}z} \qquad 0<\rho<1\,. \tag{5.1}$$

Proof. Using (2.22), we can write

$$H(q^k x, q^l y, q) = \frac{(y;q)_l}{2\pi\mathrm{i}}\frac{(q;q)_\infty}{(y;q)_\infty}\int_{\rho-\mathrm{i}\infty}^{\rho+\mathrm{i}\infty}\frac{1}{z^k(y/z;q)_l}\frac{(y/z;q)_\infty}{(z;q)_\infty}z^{-\log x/\log q}\,\mathrm{d}z$$
$$0<\rho<1\,. \tag{5.2}$$

Inserting this into

$$G(x,y,q) = y\left(\frac{H(q^2x,qy,q)}{H(qx,qy,q)} - 1\right) \tag{5.3}$$

and combining the prefactors results in (5.1). □

We again simplify the integrands, resulting in the next lemma.

Lemma 5.2. Let $0 < x, y < 1$ and $q = \mathrm{e}^{-\varepsilon}$ for $\varepsilon > 0$. Then

$$G(x,y,q) = \frac{\int_{\rho-\mathrm{i}\infty}^{\rho+\mathrm{i}\infty}\mathrm{e}^{\frac{1}{\varepsilon}\{\log(z)\log(x)+\mathrm{Li}_2(z)-\mathrm{Li}_2(y/z)\}}\frac{y}{z}\sqrt{\frac{(1-z)}{z(z-y)}}\,\mathrm{d}z}{\int_{\rho-\mathrm{i}\infty}^{\rho+\mathrm{i}\infty}\mathrm{e}^{\frac{1}{\varepsilon}\{\log(z)\log(x)+\mathrm{Li}_2(z)-\mathrm{Li}_2(y/z)\}}\sqrt{\frac{1}{z(1-z)(z-y)}}\,\mathrm{d}z}\{1+\mathrm{O}(\varepsilon)\} \tag{5.4}$$

where $y < \rho < 1$.

Proof. As in the proof of lemma 4.1, we approximate the integrands using (3.14). The exponential parts of the involved integrals, as well as the error terms, are identical to the ones in lemma 4.1 so that the proof carries over directly. □

From the work in the previous section it is clear that we will arrive at an asymptotic expression of the form

$$G(x,y,q) \sim \frac{p_0^{(1)}\varepsilon^{1/3}V(\alpha\varepsilon^{-2/3}) + q_0^{(1)}\varepsilon^{2/3}V'(\alpha\varepsilon^{-2/3})}{p_0^{(2)}\varepsilon^{1/3}V(\alpha\varepsilon^{-2/3}) + q_0^{(2)}\varepsilon^{2/3}V'(\alpha\varepsilon^{-2/3})} \tag{5.5}$$

and all that is left is to determine the coefficients involved. We get

Theorem 5.3. Let $0 < x, y < 1$ and $q = \mathrm{e}^{-\varepsilon}$ for $\varepsilon > 0$. Then

$$G(x,y,q) = \tfrac{1}{2}\left\{1 - x - y + \frac{\mathrm{Ai}'(\alpha\varepsilon^{-2/3})}{\alpha^{1/2}\varepsilon^{-1/3}\mathrm{Ai}(\alpha\varepsilon^{-2/3})}\sqrt{(1-x-y)^2 - 4xy}\right\}\{1 + \mathrm{O}(\varepsilon)\} \tag{5.6}$$

is a uniform asymptotic expansion in ε, where

$$\begin{aligned}\tfrac{4}{3}\alpha^{3/2} = {} & \log(z_m + \sqrt{d})\log(1 - z_m + \sqrt{d}) - \log(z_m - \sqrt{d})\log(1 - z_m - \sqrt{d}) \\ & + \mathrm{Li}_2(z_m - \sqrt{d}) + \mathrm{Li}_2(1 - z_m - \sqrt{d}) - \mathrm{Li}_2(z_m + \sqrt{d}) - \mathrm{Li}_2(1 - z_m + \sqrt{d})\end{aligned} \tag{5.7}$$

with

$$z_{1,2} = z_m \pm \sqrt{d} \qquad z_m = \frac{1 + y - x}{2} \qquad \text{and} \qquad d = z_m^2 - y\,. \tag{5.8}$$

Proof. We use the approximation from lemma 5.2. As the exponential part $g(z)$, as well as the path of the integral, is identical to the one in lemma 4.2, the whole argument carries over. Equation (4.14) gives the same α and β and we can again identify $V(\lambda)$ with $\mathrm{Ai}(\lambda)$. Moreover, using (4.19) we can compute the coefficients for the leading terms as

$$p_0^{(1)} = \tfrac{1}{2}(1 - x - y)\left(\frac{\alpha}{d}\right)^{1/4} \qquad q_0^{(1)} = \left(\frac{d}{\alpha}\right)^{1/4} \tag{5.9}$$

for the enumerator and

$$p_0^{(2)} = \left(\frac{\alpha}{d}\right)^{1/4} \qquad q_0^{(2)} = 0 \tag{5.10}$$

for the denominator. Inserting these into (5.5) results in (5.6). □

In [5] we computed a scaling form from the semicontinuous limit

$$G_{\mathrm{sc}}(x,y,q) = \lim_{a\to 0}\frac{1}{a}G(a^2x, y^a, q^a)\,. \tag{5.11}$$

Taking the same limit in (5.6) we recover that scaling form. We note that the uniform asymptotic expression presented here is more general and exhibits the symmetry between x and y, a feature that gets lost upon taking the semicontinuous limit as in [5]. The same scaling form has also been derived in the semicontinuous models of column-convex vesicles [6], with the only difference being different non-universal constants. However, this is the first time that the scaling form has been derived directly for the discrete model.

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